6-4 Energy Storage in Capacitors
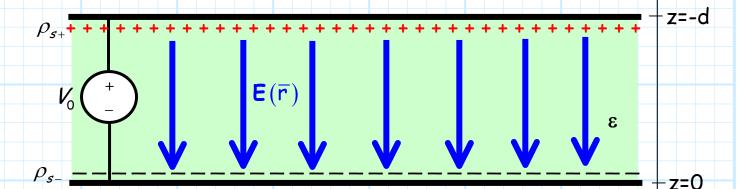
Reading Assignment: *pp. 200-201*

Q:

A: HO: Energy Storage in Capacitors

Energy Storage in Capacitors

Recall in a **parallel plate capacitor**, a surface charge distribution $\rho_{s+}(\bar{r})$ is created on **one** conductor, while charge distribution $\rho_{s-}(\bar{r})$ is created on the **other**.

Q: *How much **energy** is stored by these charges?*

A: Use the results of **Section 6-3**.

We learned that the energy **stored** by a **charge distribution** is:

$$W_e = \frac{1}{2} \iiint_V \rho_v(\bar{r}) V(\bar{r}) \; dv$$

The **equivalent** equation for **surface** charge distributions is:

$$W_e = \frac{1}{2} \iint_S \rho_s(\bar{r}) V(\bar{r}) \, ds$$

For the parallel plate capacitor, we must integrate over **both** plates:

$$W_e = \frac{1}{2} \iint_{S_+} \rho_{s+}(\bar{r}) V(\bar{r}) \, ds + \frac{1}{2} \iint_{S_-} \rho_{s-}(\bar{r}) V(\bar{r}) \, ds$$

But on the **top** plate (i.e., S_+), we know that:

$$V(z=-d) = V_0$$

while on the **bottom** (i.e., S_-):

$$V(z=0) = 0$$

Therefore:

$$W_e = \frac{V_0}{2} \iint_{S_+} \rho_{s+}(\bar{r}) \, ds + \frac{0}{2} \iint_{S_-} \rho_{s-}(\bar{r}) \, ds$$

$$= \frac{V_0}{2} \iint_{S_+} \rho_{s+}(\bar{r}) \, ds$$

But, the remaining surface integral we know to be **charge** Q:

$$Q = \iint_{S_+} \rho_{s+}(\bar{r}) \, ds$$

Therefore, we find:

$$W_e = \frac{1}{2} V_0 \, Q$$

But recall that:

$$Q = CV$$

where V is the **potential difference** between the two conductors (i.e., V =V$_0$).

Combining these **two** equations, we find:

$$
\begin{aligned}
W_e &= \frac{1}{2} V_0\, Q \\[2mm]
&= \frac{1}{2} V_0\, (C\,V) \\[2mm]
&= \frac{1}{2} C\, V^2
\end{aligned}
$$

The above equation shows that the **energy stored** within a capacitor is proportional to the product of its capacitance and the squared value of the voltage across the capacitor.

Recall that we also can determine the stored energy from the **fields** within the dielectric:

$$
W_e = \frac{1}{2} \iiint\limits_{V} \mathbf{D}(\bar{r}) \cdot \mathbf{E}(\bar{r})\, dv
$$

Since the fields within the capacitor are **approximately**:

$$
\mathbf{E}(\bar{r}) = \frac{V}{d}\,\hat{a}_z \qquad\qquad \mathbf{D}(\bar{r}) = \frac{\varepsilon V}{d}\,\hat{a}_z
$$

we find:

$$W_e = \frac{1}{2} \iiint_V \mathbf{D}(\bar{r}) \cdot \mathbf{E}(\bar{r}) \, dv$$

$$= \frac{1}{2} \iiint_V \frac{\varepsilon V^2}{d^2} \, dv$$

$$= \frac{1}{2} \frac{\varepsilon V^2}{d^2} \iiint_V \, dv$$

$$= \frac{1}{2} \frac{\varepsilon V^2}{d^2} (Volume)$$

where the volume of the dielectric is simply the plate **surface area** S time the dielectric **thickness** d:

$$Volume = S \, d$$

Resulting in the expression:

$$W_e = \frac{1}{2} \frac{\varepsilon V^2}{d^2} (S \, d)$$

$$= \frac{1}{2} \frac{\varepsilon S}{d} V^2$$

Recall, however, that the **capacitance** of a parallel plate capacitor is:

$$C = \frac{\varepsilon S}{d}$$

Therefore:

$$W_e = \frac{1}{2} \frac{\varepsilon S}{d} V^2 = \frac{1}{2} C V^2$$

The **same** result as before!